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Relativistic quantum physics with hyperbolic numbers

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Abstract

A representation of the quadratic Dirac equation and the Maxwell equations in terms of the three-dimensional universal complex Clifford algebra $\bar{\mathcal{C}}_{3,0}$ is given. The investigation considers a subset of the full algebra, which is isomorphic to the Baylis algebra. The approach is based on the two Casimir operators of the Poincaré group, the mass operator and the spin operator, which is related to the Pauli–Lubanski vector. The extension to spherical symmetries is discussed briefly. The structural difference to the Baylis algebra appears in the shape of the hyperbolic unit, which plays an integral part in this formalism.

1. Introduction

In the same way as complex numbers are associated with the Euclidean geometry, the two other systems of bidimensional hypercomplex numbers can be associated with geometries of physical relevance. The parabolic numbers can be associated with the Galileo group and the hyperbolic numbers with the Lorentz group of special relativity [1,2].

The Hamilton quaternions as well as the bidimensional systems are included in the more general Clifford algebras [3–6]. They can represent the Euclidean geometry because their invariant quantity is an algebraic quadratic form as well as the Euclidean and pseudo-Euclidean invariants.

The hyperbolic numbers offer the possibility to represent the four-component Dirac spinor as a two-component hyperbolic spinor. Hucks has shown [7] that the Lorentz group is equivalent to the hyperbolic unitary group and that the operations of C, P, and T on Dirac spinors are closely related to the three types of complex conjugation that exist when both hyperbolic and ordinary imaginary units are present.

Since the relativistic spin group is representable as an unitary group, the special linear group, which is normally used to represent the relativistic spin, has to be considered as an unitary group as well. Porteous [8, 9] proves the unitarity of the special linear group with the help of the double field, which corresponds to the null basis representation of the hyperbolic numbers.

There are various applications of hyperbolic numbers. They have been applied by Reany [10] to 2nd-order linear differential equations. A function theory

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for hyperbolic numbers has been presented by Motter and Rosa [11]. Extensions to an n-dimensional space have been given by several authors [12–15], including an analysis of hyperbolic Fourier transformations [16]. The functional calculus of hyperbolic numbers is also covered by the more general approach of superanalysis (see, e.g., Khrennikov [17]). Considerations of a non-relativistic hyperbolic Hilbert space with respect to the Born formula have been given by Kocik [18], Khrennikov [19,20], and Rochon and Tremblay [21]. Xuegang et al. investigated the Dirac wave equation, Clifford algebraic spinors, a hyperbolic Hilbert space, and the hyperbolic spherical harmonics in hyperbolic spherical polar coordinates [22–24]. The hypercomplex numbers, and the geometries generated by these numbers, have been investigated by Catoni et al. [25]. Further applications of hyperbolic numbers, including e.g., their application to general relativity by Kunstatter et al., can be found in [26–33].

It has been shown by Baylis and Jones [34] that a $\boldsymbol{R}_{3,0}$ Clifford algebra has enough structure to describe relativity as well as the more usual $\boldsymbol{R}_{1,3}$ Dirac algebra or the $\boldsymbol{R}_{3,1}$ Majorana algebra. Baylis represents relativistic space–time points as paravectors and applies these paravectors to electrodynamics [35]. The approach to relativity in terms of $\boldsymbol{R}_{1,3}$ has been investigated by Hestenes [3,36,37] or, e.g., by Gull et al. [38]. An overview of the structural differences in the above low-dimensional Clifford algebras is given by Dimakis [39] in the context of a general spinor representation within Clifford algebras.

The approach used in this work is congruent to Baylis paravectors. However, hyperbolic numbers are included in the algebra. This corresponds actually to a hyperbolic complexification of the Baylis algebra. According to Porteous [9] the resulting algebra is isomorphic to the three-dimensional universal complex Clifford algebra $\bar{\boldsymbol{C}}_{3,0}$. The correspondence to the Baylis algebra is given by the restriction to a subset of the algebra. This restriction can be justified in the hyperbolic Hilbert space by the hermiticity of the Poincaré mass operator [40].

This work is an extension of [41]. It introduces beside the Poincaré mass operator an analogous operator for the spin, which is related to the Pauli–Lubanski vector. In addition, the approach is extended to spherical symmetries. The mathematical background of the presented approach with respect to the classification of Clifford algebras is not fully explained in [40,41]. This will be improved in this work based on the general overview given by Porteous [8,9]. The notation of Porteous has been adopted in many cases.

2. Hyperbolic numbers

Vector spaces can be defined over the commutative ring of hyperbolic numbers $z \in \boldsymbol{H}$

$$z = x + iy + jv + ijw, \quad x, y, v, w \in \boldsymbol{R}, \tag{1}$$

where the complex unit i and the hyperbolic unit j have the properties

$$i^2 = -1, \qquad j^2 = 1. \tag{2}$$

The hyperbolic numbers defined in this way are a commutative extension of the complex numbers to include new roots of the polynomial equation $z^2 - 1 = 0$. In the terminology of Clifford algebras they are represented by $\bar{\boldsymbol{C}}_{1,0}$, i.e., they correspond to the universal one-dimensional complex Clifford algebra (the notation follows Porteous [9]).

Beside the grade involution, two anti-involutions play a major role in the description of Clifford algebras and their structure, conjugation and reversion. Conjugation changes the sign of the complex and the hyperbolic unit

$$\bar{z} = x - iy - jv + ijw. \tag{3}$$

The hyperbolic numbers with conjugation are equivalent to the double field ${}^2\bar{\boldsymbol{C}}^{\sigma}$ of Porteous, where σ and the bar symbol denote swap and complex conjugation. The notation of Porteous has the advantage that it can clearly specify, whether the double field is defined over the real ${}^2\boldsymbol{R}$, complex ${}^2\boldsymbol{C}$, or quaternionic numbers ${}^2\boldsymbol{Q}$. If necessary the notation used in this work is extended to $\boldsymbol{H}(\boldsymbol{R})$, $\boldsymbol{H}(\boldsymbol{C}) = \boldsymbol{H}$, or $\boldsymbol{H}(\boldsymbol{Q})$, where $\boldsymbol{H}(\boldsymbol{R})$ and $\boldsymbol{H}(\boldsymbol{Q})$ correspond to the universal Clifford algebras $\boldsymbol{R}_{1,0}$ and $\boldsymbol{R}_{0,3}$, respectively.

With respect to the Clifford conjugation the square of the hyperbolic number can be calculated as

$$z\bar{z} = x^2 + y^2 - v^2 - w^2 + 2ij(xw - yv), \tag{4}$$

i.e., in general the square of a hyperbolic number is not a real number.

Beside the conjugation, the second important anti-involution is the reversion, which changes only the sign of the complex unit

$$z^\dagger = x - iy + jv - ijw. \tag{5}$$

Anti-involutions reverse the ordering in the multiplication, e.g., $(ab)^\dagger = b^\dagger a^\dagger$. This becomes important when non-commuting elements of an algebra are considered. In physics, reversion is denoted as Hermitian conjugation. Note, that in [40] it has been suggested to relate hermiticity in the physical sense to the conjugation anti-involution.

3. Hyperbolic algebra

Consider a hyperbolic vector with the coordinates $z^\mu = (z^0, z^i) \in \bar{\boldsymbol{H}}^{3,1}$. The bar symbol indicates that the vector has the signature $(3,1)$ in the Hermitian product with respect to conjugation, i.e., $z_\mu \bar{z}^\mu = z_0 \bar{z}^0 - z_i \bar{z}^i$. The vector can be represented in terms of a Clifford algebra as

$$Z = z^\mu e_\mu. \tag{6}$$

The basis elements $e_\mu = (e_0, e_i)$ include the unity and the Pauli algebra multiplied by the hyperbolic unit j

$$e_\mu = (1, j\sigma_i). \tag{7}$$

The pseudoscalar of the hyperbolic algebra, which will appear throughout this work, is defined as

$$I = e_0 \bar{e}_1 e_2 \bar{e}_3 = ij. \tag{8}$$

Important for the further analysis is the behaviour of the above hypercomplex units under conjugation and reversion. In Table 1 it is displayed whether the sign of the unit is changed or not under the considered operation. For completeness the graduation as an example of an important involution is displayed. An involution does not change the ordering in a product, i.e., $\widehat{ab} = \hat{a}\hat{b}$. The graduation is used to identify the even elements of a Clifford algebra. A certain subset of these elements defines the spin group of the considered Clifford algebra [9].

Table 1
Effect of conjugation, reversion, and graduation on the used hypercomplex units

a	$\bar{a}$	$a^\dagger$	$\hat{a}$
e_i	−	+	−
σ_i	+	+	+
I	+	−	+
i	−	−	−
j	−	+	−

Adding the hyperbolic unit to the Pauli algebra corresponds in fact to a hyperbolic complexification. In the terminology of Clifford algebras the complexification is leading to the isomorphisms

$$\boldsymbol{R}_{3,0} \otimes \bar{\boldsymbol{H}}(\boldsymbol{R}) \simeq \bar{\boldsymbol{H}}(\boldsymbol{R})_{3,0} \simeq \boldsymbol{H}(2) \simeq \bar{\boldsymbol{C}}_{3,0}. \tag{9}$$

The full structure therefore corresponds to the universal three-dimensional complex Clifford algebra. The representation of the Clifford algebra in terms of matrices is given by $\boldsymbol{H}(2)$, the algebra of hyperbolic 2×2 matrices. This representation will be assumed in some of the equations within this work.

The vector Z of Eq. (6) has sixteen real dimensions. In [40] it has been shown that the four-dimensional real Minkowski vector can be considered as the magnitude of Z, if the square is restriced to real numbers, i.e., $Z\bar{Z} \in \boldsymbol{R}$. Based on the assumption that only such vectors are of physical relevance, the following investigation is restricted to the real four-dimensional Minkowski space, and the remaining phase contributions are neglected. A Minkowski vector $x^\mu \in \boldsymbol{R}^{3,1}$ is expressed in the above algebra as $X = x^\mu e_\mu$.

The spatial vector contributions can be written explicitly in the following representation

$$X = x^0 + j\boldsymbol{x}, \tag{10}$$

where $\boldsymbol{x} = x^i \sigma_i$. Using the Pauli matrices as the explicit representation of σ_i, the vector X can be expressed in terms of a hyperbolic 2×2 matrix according to

$$X = \begin{pmatrix} x^0 + jx^3 & jx^1 - ijx^2 \\ jx^1 + ijx^2 & x^0 - jx^3 \end{pmatrix}. \tag{11}$$

The scalar product of two vectors can be defined as

$$X \cdot Y = \frac{1}{2}(X\bar{Y} + Y\bar{X}). \tag{12}$$

The wedge product is given as

$$X \wedge Y = \frac{1}{2}(X\bar{Y} - Y\bar{X}). \tag{13}$$

The wedge product corresponds to a so-called biparavector, which can be used, e.g., for the description of the electromagnetic field or the relativistic angular momentum (see also Baylis [35]).

The basis elements of the relativistic $\bar{\boldsymbol{C}}_{3,0}$ paravector algebra can be considered as the basis vectors of the relativistic vector space. These basis elements form a non-Cartesian orthogonal basis with respect to the scalar product defined in Eq. (12)

$$e_\mu \cdot e_\nu = g_{\mu\nu}, \tag{14}$$

where $g_{\mu\nu}$ is the metric tensor of the Minkowski space.

As an example of the paravector algebra the energy–momentum vector of a free classical pointlike particle, moving with velocity $\boldsymbol{v}$ relative to the observer, is expressed in the Pauli algebra notation. The relativistic momentum vector for this particle can be written as

$$P = \frac{E}{c} + j\boldsymbol{p} = mc\exp(j\boldsymbol{\xi}), \tag{15}$$

with c denoting the velocity of light, $\boldsymbol{\xi}$ the rapidity, E the energy and $\boldsymbol{p}$ the momentum of the particle. The rapidity is defined as $\tanh\xi = v/c = pc/E$, where $\xi = |\boldsymbol{\xi}|$ and $p = |\boldsymbol{p}|$. Rapidity and momentum point into the same direction $\boldsymbol{n} = \boldsymbol{v}/|\boldsymbol{v}|$ as the velocity. In the following c and $\hbar$ will be set equal to one. In quantum mechanics the momentum is replaced in coordinate space by the operators $p^\mu = i\partial^\mu$.

4. Lorentz transformations

Porteous [9] shows that the general linear group can be considered as an unitary group. The unitarity is related to the double field together with an appropriate correlation. For the complex linear group this is a ${}^2\bar{\boldsymbol{C}}^\sigma$-correlation. The complex double field ${}^2\bar{\boldsymbol{C}}^\sigma$, with swap σ and conjugation, corresponds to the null basis representation of the hyperbolic numbers with conjugation $\bar{\boldsymbol{H}}$ as given in Eqs. (1) and (3). In addition, there is an isomorphism between the general linear group and the hyperbolic unitary group $GL(n,\boldsymbol{C}) \simeq U(n,\boldsymbol{H})$, and in particular, for the special groups in two dimensions, i.e., $SL(2,\boldsymbol{C}) \simeq SU(2,\boldsymbol{H})$ (see also Hucks [7]). The unitarity of the group $U(n,\boldsymbol{H})$ is understood with respect to a $\bar{\boldsymbol{H}}$-correlation.

The group $SU(2,\boldsymbol{H})$ corresponds to the spin group of $SO(3,1,\boldsymbol{R})$ and its elements can be used to express rotations and boosts of the paravectors defined in the last section. The rotation of a paravector can be expressed as [35]

$$X \to X' = RXR^\dagger. \tag{16}$$

For the boosts one finds the transformation rule

$$X \to X' = BXB^\dagger. \tag{17}$$

The rotations and boosts are given as

$$R = \exp(-i\boldsymbol{\theta}/2), \qquad B = \exp(j\boldsymbol{\xi}/2). \tag{18}$$

Based on the Pauli matrices an explicit matrix representation of the boost operator B can be given, e.g., for a boost in the direction of the x-axis one finds

$$B_1 = \begin{pmatrix} \cosh\xi_1/2 & j\sinh\xi_1/2 \\ j\sinh\xi_1/2 & \cosh\xi_1/2 \end{pmatrix}. \tag{19}$$

The boosts are invariant under reversion $B^\dagger = B$, whereas the conjugated boost corresponds to the inverse $\bar{B} = B^{-1}$. For rotations reversion and conjugation correspond both to the inverse $R^\dagger = \bar{R} = R^{-1}$. This relationship indicates that in non-relativistic physics Hermitian operators can be defined either with respect to reversion or conjugation.

Boosts and rotations can be combined to form the Lorentz transformation

$$X \to X' = LXL^\dagger, \tag{20}$$

which can be expressed in terms of its infinitesimal generators as

$$L = \exp\bigl(-i\theta^i J_i - i\xi^i K_i\bigr). \tag{21}$$

From these equations the infinitesimal generators of a Lorentz transformation can be identified as

$$\boldsymbol{J} = \boldsymbol{\sigma}/2, \qquad \boldsymbol{K} = ij\boldsymbol{\sigma}/2. \tag{22}$$

One can show that the generators satisfy the Lie algebra of the Lorentz group $SO(3,1,\boldsymbol{R})$. The Lorentz transformations can be expressed also with relativistic second rank tensors

$$L = \exp\bigl(-iS_{\mu\nu}\omega^{\mu\nu}/2\bigr), \tag{23}$$

where the spin angular momentum is defined in terms of the wedge product as

$$e_\mu \wedge e_\nu = 2S_{\mu\nu}. \tag{24}$$

The elements of the spin angular momentum therefore represent planes in space–time that are formed by the basis elements of the algebra.

The following example shows how coordinate vector, momentum vector, orbital angular momentum and spin angular momentum can be related to each other in the paravector algebra

$$X\bar{P} = x_\mu p^\mu - iS_{\mu\nu}L^{\mu\nu}, \tag{25}$$

where $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ corresponds to the relativistic orbital angular momentum.

5. Poincaré mass operator

With the above vector representation the Poincaré mass operator can be introduced as a product of a momentum vector and its conjugated counterpart

$$M^2 = P\bar{P}. \tag{26}$$

The explicit form of the mass operator is obtained by a multiplication of the basis matrices. The mass operator can be separated into a spin dependent and a spin independent contribution

$$P\bar{P} = p_\mu p^\mu - i\sigma_{\mu\nu}p^\mu p^\nu, \tag{27}$$

where the spin term is given by

$$\sigma_{\mu\nu} = \begin{pmatrix} 0 & -ij\sigma_1 & -ij\sigma_2 & -ij\sigma_3 \\ ij\sigma_1 & 0 & \sigma_3 & -\sigma_2 \\ ij\sigma_2 & -\sigma_3 & 0 & \sigma_1 \\ ij\sigma_3 & \sigma_2 & -\sigma_1 & 0 \end{pmatrix}. \tag{28}$$

Since the spin contribution is anti-symmetric, the last term in Eq. (27) is in this case zero. The spin structure becomes important when interactions are introduced by the minimal substitution of the momentum operators. The anti-symmetric contribution $\sigma_{\mu\nu}$ corresponds to the wedge product of Eq. (24), which gives the analogous relation $\sigma_{\mu\nu} = 2S_{\mu\nu}$.

The basic fermion equation is introduced as an eigenvalue equation of the mass operator. With the hyperbolic algebra defined above the equation can be written as

$$M^2\psi(x) = m^2\psi(x). \tag{29}$$

The wave function $\psi(x)$ has the general structure

$$\psi(x) = \varphi(x) + j\chi(x), \tag{30}$$

where $\varphi(x)$ and $\chi(x)$ can be represented as two-component spinor functions (see Hucks [7]). They depend on the four space–time coordinates x^μ.

6. Poincaré spin operator

In analogy to the mass operator a spin operator can be introduced. The spin operator corresponds to the second Casimir operator of the Poincaré group, which describes a system that is invariant under relativistic translations and rotations. The basic equation for the spin operator can be defined as

$$S^2\psi(x) = -s(s+1)\psi(x), \tag{31}$$

where the square of the spin operator corresponds to

$$S^2 = \frac{W\bar{W}}{m^2}. \tag{32}$$

The operator W denotes the Pauli–Lubanski vector, which can be expressed in terms of the complex Clifford algebra as

$$W = -\boldsymbol{J}\cdot\boldsymbol{p} - j\big(\boldsymbol{J}p^0 + \boldsymbol{K}\times\boldsymbol{p}\big). \tag{33}$$

The spatial spin operator can be derived from the Pauli–Lubanski vector by projections (see Michel [42] or Wightman [43])

$$\boldsymbol{S} = \frac{1}{m}W\cdot\boldsymbol{n}. \tag{34}$$

The projection vectors n^μ are an arbitrary set of four orthogonal vectors satisfying the relation $n^\mu\cdot n^\nu = g^{\mu\nu}$. The spin operator takes the following form

$$\boldsymbol{S} = \frac{1}{m}\left(\boldsymbol{J}p^0 + \boldsymbol{K}\times\boldsymbol{p} - (\boldsymbol{J}\cdot\boldsymbol{p})\frac{\boldsymbol{p}}{p^0+m}\right), \tag{35}$$

if the set of projection vectors is chosen as

$$n^0 = m^{-1}\big(p^0, p^k\big),$$
$$n^i = m^{-1}\left(p^i, m\delta^{ik} + \frac{p^i p^k}{p^0+m}\right). \tag{36}$$

The eigenstates of the spin operator can be introduced as eigenvectors of the squared spin operator and of the z-component $S_z = S_3$

$$S^2|sm_s\rangle = -s(s+1)|sm_s\rangle,$$
$$S_z|sm_s\rangle = m_s|sm_s\rangle. \tag{37}$$

To find an explicit representation of these eigenstates one has to consider that the spin operator in the above form can be obtained also with a boost acting on the operator vector $\boldsymbol{J}$

$$\boldsymbol{S} = B\boldsymbol{J}\bar{B}. \tag{38}$$

Therefore, the relativistic spinor can be related to the non-relativistic Pauli spinor by a boost

$$|sm_s\rangle = B\chi_{m_s} = u(\boldsymbol{p}, m_s). \tag{39}$$

If the rapidity in the boost (see Eq. (18)) is expressed in terms of the particle momentum, one finds that the boosted Pauli spinor corresponds to the hyperbolic representation of the Dirac spinor

$$u(\boldsymbol{p}, m_s) = \sqrt{\frac{p^0+m}{2m}}\left(1+\frac{j\boldsymbol{p}}{p^0+m}\right)\chi_{m_s}. \tag{40}$$

The anti-particle spinor can be derived from the particle spinor, which is multiplied by the hyperbolic unit $v(\boldsymbol{p}, m_s) = ju(\boldsymbol{p}, m_s)$. The spinors can be combined with the Hilbert space state for the momentum $|p^\mu\rangle$ to form the plane wave expansion

$$\psi(x) = \sum_{m_s}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2p^0}\big(u(\boldsymbol{p}, m_s)e^{-ip_\mu x^\mu}b(p, m_s) + v(\boldsymbol{p}, m_s)e^{ip_\mu x^\mu}\bar{d}(p, m_s)\big), \tag{41}$$

which is a general solution of the Poincaré mass operator and the Poincaré spin operator (31).

The relativistic wave function is an element of the spinor space, which is a minimal left ideal in the terminology of Clifford algebras. The elements of a minimal left ideal have rank 1 and therefore they can be represented as column vectors $\psi^i(x) \in \bar{\boldsymbol{H}}^2$. The transformation rule of the spin operator in Eq. (38) is in contrast to Eq. (17). The above rule is used for relativistic operators acting in the spinor space. The transformation rule is related to the $\bar{\boldsymbol{H}}$-correlation, which maps the elements of the spinor space to their dual space. Since the Poincaré mass operator and the Poincaré spin operator are operators in the spinor space they have to obey the same transformation law. This is the reason why, e.g., the mass operator is defined as $P\bar{P}$ and not as $PP^\dagger$.

7. Massless particles

Massless particles like neutrinos are normally described in terms of helicity states. It is shown that massless particles can also be described with boosted Pauli spinors. Representing particles in terms of Pauli spinors corresponds to a decoupling of the polarization axis and the direction of momentum. In explicit calculations this decoupling is leading to ambiguities in the coupling of angular momenta of multiple particles. Therefore, e.g., in calculations of scattering amplitudes, the helicity basis is the best choice, even in a non-relativistic scheme.

From the theoretical point of view, massless particles that are represented by Pauli spinors provide a direct analogy to the description of the last section. What has to be shown is that this representation is conform with the Poincaré group in the massless case

$$M^2\psi(x) = 0. \tag{42}$$

If the polarization axis for a massive particle is chosen, e.g., as the z-axis, the momentum can take any direction without any restriction. This is not the case for massless particles. The spatial momentum can never be perpendicular to the chosen polarization axis. The spatial symmetry of the momentum is therefore broken, which will become apparent in the following equations.

The spinor will be defined within the little group of a standard vector [44]. Since massless particles are moving with the velocity of light they have no rest frame. Therefore, the standard frame will be defined as the system in which the momentum is directed along the polarization axis. If the particle is polarized in the direction of the z-axis the positive-energy standard vector is given as

$$p^\mu = \big(|p^3|, 0, 0, p^3\big). \tag{43}$$

In this description the helicity can be either positive or negative. The states are always characterized by the two possible polarizations corresponding to the chosen polarization axis. The components of the Pauli–Lubanski vector in the standard frame are

$$\begin{aligned}
W^0 &= -p^3 J^3,\\
W^1 &= -|p^3|\big(J^1 + K^2\big),\\
W^2 &= -|p^3|\big(J^2 - K^1\big),\\
W^3 &= -|p^3|J^3.
\end{aligned} \tag{44}$$

The operators W^0 and W^3 are linear dependent and can be represented by J^3. The three generators J^3, W^1 and W^2 satisfy the Lie algebra of E_2, the Euclidean group in two dimensions, which defines the little group of the $m^2 = 0$ representation.

An arbitrary momentum vector p^μ is obtained with a boost perpendicular to the z-axis

$$p^\mu = \left(\left|p^3\right| \cosh\xi, \left|p^3\right| n^1 \sinh\xi, \left|p^3\right| n^2 \sinh\xi, p^3\right). \tag{45}$$

The unit vector $n^i_\perp = (n^1, n^2, 0)$ characterizes the direction of the boost. With the perpendicular momentum vector $p^i_\perp = (p^1, p^2, 0)$ the rapidity can be defined by the relation $\tanh\xi = p_\perp / p^0$, where $p_\perp = |\boldsymbol{p}_\perp|$. Using $\boldsymbol{\xi}_\perp = \boldsymbol{n}_\perp \xi$ the boost can be written in the form $B = \exp(j\boldsymbol{\xi}_\perp/2)$. The momentum contribution parallel to the polarization will be denoted as $p^i_\parallel = (0, 0, p^3)$, with $p_\parallel = |\boldsymbol{p}_\parallel|$.

In the spinor representation the generators $\boldsymbol{J} = \boldsymbol{\sigma}/2$ and $\boldsymbol{K} = ij\boldsymbol{\sigma}/2$ can be inserted into Eq. (44). Then one finds in the standard frame and therefore in all frames $W\bar{W} = 0$, i.e., the spin is given in the degenerate spin $s = 0$ representation of E_2. The basis vectors are chosen as eigenvectors of J_3 with the eigenvalues $m_s = \pm 1/2$. They can be represented with the Pauli spinor χ_{m_s}. A general spinor can be calculated using Eq. (39) with the boost parameters defined above

$$u(\boldsymbol{p}, m_s) = \sqrt{\frac{p^0 + p_\parallel}{2p_\parallel}} \left(1 + \frac{j\boldsymbol{p}_\perp}{p^0 + p_\parallel}\right) \chi_{m_s}. \tag{46}$$

The antiparticle spinor is obtained if the above expression is multiplied by the hyperbolic unit.

A set of four orthogonal projection vectors can be introduced to derive the spin operator from the Pauli–Lubanski vector

$$\begin{aligned} n^0 &= p_\parallel^{-1}\left(p^0, p^k_\perp\right), \\ n^i &= p_\parallel^{-1}\left(p^i_\perp, p_\parallel \delta^{ik} + \frac{p^i_\perp p^k_\perp}{p^0 + p_\parallel}\right). \end{aligned} \tag{47}$$

The spin operator is then defined as

$$\boldsymbol{S} = \frac{1}{p_\parallel} W \cdot \boldsymbol{n}. \tag{48}$$

The spin operator corresponds again to a boosted vector of the angular momentum generators $\boldsymbol{J}$. Explicitly written one finds

$$\boldsymbol{S} = \frac{1}{p_\parallel}\left(\boldsymbol{J} p^0 + \boldsymbol{K} \times \boldsymbol{p}_\perp - (\boldsymbol{J} \cdot \boldsymbol{p}_\perp)\frac{\boldsymbol{p}_\perp}{p^0 + p_\parallel}\right). \tag{49}$$

As mentioned above only the third component $S_z = S_3$ of the spin operator is relevant within E_2. For this component the last term in the above equation is zero. The action of the spin operators on the spinor can be summarized in the relations

$$\begin{aligned} S^2|0m_s\rangle &= 0, \\ S_z|0m_s\rangle &= m_s|0m_s\rangle, \end{aligned} \tag{50}$$

where the square of the spin operator corresponds to $S^2 = W\bar{W}/p^2_\parallel$. The plane wave expansion is formally equivalent to Eq. (41), but the spinors $u(\boldsymbol{p}, m_s)$ and $v(\boldsymbol{p}, m_s)$ have to be replaced with the specific $m^2 = 0$ form given above. From these equations it follows that

$$S^2\psi(x) = 0. \tag{51}$$

In fact, massless particles are also spinless particles. However, they can be described by a non-zero momentum and a polarization comparable to the massive particles.

8. Maxwell equations

The Maxwell equations can be derived from an eigenvalue equation of the mass operator, where the mass operator is now acting on a vector field

$$M^2 A(x) = 0. \tag{52}$$

The equation can be expressed with the electromagnetic fields according to

$$\begin{aligned} P\bar{P}A = &-\nabla \cdot \boldsymbol{E} - \partial^0 C \\ &+ ij\nabla \cdot \boldsymbol{B} \\ &- j\left(\nabla \times \boldsymbol{B} - \partial^0 \boldsymbol{E} - \nabla C\right) \\ &- i\left(\nabla \times \boldsymbol{E} + \partial^0 \boldsymbol{B}\right) = 0. \end{aligned} \tag{53}$$

This expression is obtained if one evaluates $\bar{P}A(x)$, inserts the usual definitions for the electromagnetic fields, and then multiplies the resulting terms by the operator P. If $P\bar{P}$ is calculated first, Eq. (52) reduces to the wave operator acting on the vector potential giving zero. Both forms are equivalent in the Lorentz gauge.

In Eq. (53) the four homogeneous Maxwell equations are included. The calculation provides two additional terms depending on

$$C(x) = \partial_\mu A^\mu(x). \tag{54}$$

These terms disappear in the Lorentz gauge.

9. Photon plane wave states

In this section a plane wave expansion for free photon fields is derived. The techniques developed for massless fermions will be applied. In this representation the polarization vector of the photon corresponds to a generalization of the Pauli spinor. Again it is mentioned that the helicity basis has advantages in explicit calculations.

The transformation properties of the vector components $A^\mu(x)$ can be understood in terms of 4×4 transformation matrices acting on four-component vectors

$$x^\mu \to x^{\mu\prime} = (L)^\mu{}_\nu x^\nu. \tag{55}$$

The transformation matrices can be derived from the generators J_i and K_i according to Eq. (21). The third components of the generators are

$$(J_3)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$(K_3)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{56}$$

The results of Section 7 will be used in the following. The standard frame is the system in which the momentum is directed along the polarization axis. The Pauli–Lubanski vector in the standard frame is given by Eq. (44), where the generators now have to be replaced by the generators of Eq. (56), including the remaining generator components.

The two eigenstates of J_3 with eigenvalues $m_s = \pm 1$ are given as usual as

$$\chi^\mu_\pm = \frac{\mp\epsilon^\mu_1 - i\epsilon^\mu_2}{\sqrt{2}}, \tag{57}$$

where ϵ^μ_1 and ϵ^μ_2 are unit vectors in the direction of the x- and y-axis. In the standard frame one finds that the squared spin operator acting on the eigenstates is zero

$$(W\bar{W})^\mu{}_\nu \chi^\nu_{m_s} = 0. \tag{58}$$

General eigenstates of the spin operator are obtained again by a boost of the eigenstate from the standard frame of the particle $|0m_s\rangle = e^\mu(\boldsymbol{p}, m_s) = (B)^\mu{}_\nu \chi^\nu_{m_s}$. As in Section 7 the direction of the boost is perpendicular to the polarization axis, i.e., $B = \exp(-i\xi^i_\perp K_i/2)$. The boost is performed with the generators given in Eq. (56). The calculation is leading to the explicit form of the polarization vector

$$e^\mu(\boldsymbol{p}, m_s) = \left(\frac{p^{m_s}}{p_\parallel}, \chi^i_{m_s} + \frac{p^i_\perp p^{m_s}}{p_\parallel(p^0 + p_\parallel)}\right), \tag{59}$$

where $p^\pm = (\mp p^1 - ip^2)/\sqrt{2}$. For the eigenstates one finds the relations

$$S^2|0m_s\rangle = 0,$$
$$S_z|0m_s\rangle = m_s|0m_s\rangle. \tag{60}$$

The spin operator $S_z = S_3$ is defined as in Eq. (48) with the appropriate generators for $\boldsymbol{J}$ and $\boldsymbol{K}$. The above equations are equivalent to Eq. (50) except for the different eigenvalues. For photons one finds $m_s = \pm 1$. The plane wave expansion of the free photon field is given as

$$A(x) = \sum_{m_s} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2p^0} e^\mu(\boldsymbol{p}, m_s) \times e_\mu\left(e^{-ip_\mu x^\mu} a(p, m_s) + e^{ip_\mu x^\mu} \bar{a}(p, m_s)\right). \tag{61}$$

The plane wave satisfies the relation

$$S^2 A(x) = 0. \tag{62}$$

The coordinate vector $\chi^\mu_{m_s} \in \bar{\boldsymbol{C}}^{3,1}$ plays in this context the role of the Pauli spinor. It can be considered as an element of the minimal left ideal with respect to the transformations induced by the generators of Eq. (56). This is indicated in the above formulas by the tensor indices. The indices can be omitted, e.g., $e(\boldsymbol{p}, m_s) = B\chi_{m_s}$, to provide a representation free notation.

10. Quadratic Dirac equation

Electromagnetic interactions can be introduced with the minimal substitution of the momentum operator. The mass operator of Eq. (26) transforms into

$$M^2 = \big(P - eA(x)\big)\big(\bar{P} - e\bar{A}(x)\big), \tag{63}$$

and is now invariant under local gauge transformations. This mass operator can be inserted into Eq. (29). If Pauli matrices and electromagnetic fields are expressed with the anti-symmetric tensor $\sigma_{\mu\nu}$ given in Eq. (28) and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ one finds

$$\left((p - eA)_\mu (p - eA)^\mu - \frac{e}{2}\sigma_{\mu\nu} F^{\mu\nu} - m^2\right)\psi(x) = 0. \tag{64}$$

This is the quadratic Dirac equation. Using the hyperbolic algebra it can be represented as a 2×2 matrix equation, whereas conventionally the quadratic Dirac equation is a 4×4 matrix equation. The spinor function $\psi(x)$ used for the mass operator has a two-component structure, whereas in the Dirac equation $\psi(x)$ corresponds to a four-component spinor.

One finds in both cases the same two coupled differential equations. For the hyperbolic algebra one can derive from Eq. (64)

$$\begin{aligned}&\left((p - eA)_\mu (p - eA)^\mu - eijE^i\sigma_i + eB^i\sigma_i - m^2\right)\\ &\quad\times \psi(x) = 0.\end{aligned} \tag{65}$$

In the Dirac theory the spin tensor is defined according to $\sigma_{\mu\nu} = i/2[\gamma_\mu, \gamma_\nu]$. Using this tensor the Dirac form of Eq. (65) can be written as

$$\begin{aligned}&\left((p - eA)_\mu (p - eA)^\mu - eiE^i\alpha_i + eB^i\sigma_i - m^2\right)\\ &\quad\times \psi(x) = 0.\end{aligned} \tag{66}$$

Comparing this equation with Eq. (65) one observes that in both cases the term including the electric field is the only term which couples the components of the spinor. In the mass operator equation the coupling term is proportional to $j\boldsymbol{\sigma}$, in the quadratic Dirac equation the term corresponds to $\boldsymbol{\alpha} = \gamma_5\boldsymbol{\sigma}$. The Dirac representation of γ_5 and j have the same effect on the spinor, a swap of the spinor components. For the Poincaré mass operator one finds

$$j\psi(x) = \chi(x) + j\varphi(x), \tag{67}$$

whereas in the Dirac representation the corresponding relation is given as

$$\gamma_5\psi(x) = \begin{pmatrix} \chi(x) \\ \varphi(x) \end{pmatrix}. \tag{68}$$

In the hyperbolic formalism the terms proportional to the hyperbolic unit include the differential equation of the lower component, the other terms describe the differential equation of the upper component. Conventionally, the coupled differential equations for upper and lower components are separated by the matrix structure.

11. Orbital angular momentum and single particle potentials

The hyperbolic numbers can be used also for the description of the orbital angular momentum, which will be shown in this section (compare with Xuegang [22]). A spacelike relativistic vector x^μ can be parametrized in relativistic spherical coordinates as

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \rho \sinh\xi \\ \rho\cosh\xi \sin\theta\cos\phi \\ \rho\cosh\xi\sin\theta\sin\phi \\ \rho\cosh\xi\cos\theta \end{pmatrix}. \tag{69}$$

The time coordinate is given as $\rho\sinh\xi$, where $\rho > 0$. In the limit of $\xi \to 0$ the vector reduces to a non-relativistic vector in spherical coordinates.

Based on the Lorentz transformation given in Eqs. (55) and (56) the above vector can be obtained from a standard vector $x^\mu = (0, 0, 0, \rho)$ with the following transformation

$$L(\theta, \phi, \xi) = \exp(-iJ_3\phi)\exp(-iJ_2\theta)\exp(-iK_3\xi). \tag{70}$$

For an irreducible group representation of the orbital angular momentum the relation between the generators of boosts and rotations is assumed to be the same as for the spin $(\frac{1}{2}, 0)$ representation of the Lorentz group used earlier in this work

$$\boldsymbol{K} = ij\boldsymbol{J}. \tag{71}$$

The boost generator in Eq. (70) can then be replaced by the third component of the rotation generator

$$L(\theta, \phi, \xi) = \exp(-iJ_3\phi)\exp(-iJ_2\theta)\exp(jJ_3\xi). \tag{72}$$

The orbital angular momentum will be denoted by $\boldsymbol{L}$ in contrast to the spin angular momentum $\boldsymbol{S}$. In the following equations $\boldsymbol{J}$ will be specialized to $\boldsymbol{J} = \boldsymbol{L}$. The $(l, 0)$ representation of the Lorentz group provides the following equations for the irreducible states

$$L^2|lm_l\rangle = -l(l+1)|lm_l\rangle,$$

$$L_z|lm_l\rangle = m_l|lm_l\rangle. \tag{73}$$

Again the third component is denoted as $L_z = L_3$. In this basis the transformation of Eq. (70) is represented as

$$D^l_{m_l' m_l}(\theta,\phi,\xi) = \langle lm_l'|e^{-iL_3\phi}e^{-iL_2\theta}e^{jL_3\xi}|lm_l\rangle$$
$$= e^{-im_l'\phi}d^l_{m_l' m_l}(\theta)e^{jm_l\xi}. \tag{74}$$

Compared to the non-relativistic case the relativistic rotation matrices are extended by the additional hyperbolic phase factor $e^{jm_l\xi}$.

An application of these functions can be the solution of the Poincaré mass operator with appropriate model potentials. A relativistic generalization of the $1/|\boldsymbol{x}|$ central potential is suggested, which could be used to describe an electron moving in the potential of a nucleus

$$eA^\mu(x) = -\frac{Z\alpha}{\rho}\epsilon^\mu(x). \tag{75}$$

The polarization vector corresponds to the unit vector of the ξ-coordinate

$$\epsilon^\mu(x) = \frac{1}{\rho}\frac{\partial}{\partial\xi}x^\mu(\rho,\theta,\phi,\xi)$$
$$= \begin{pmatrix} \cosh\xi \\ \sinh\xi\sin\theta\cos\phi \\ \sinh\xi\sin\theta\sin\phi \\ \sinh\xi\cos\theta \end{pmatrix}. \tag{76}$$

In the static limit $\xi \to 0$ this potential reduces to the $1/|\boldsymbol{x}|$ potential.

Some general remarks on the solution procedure will be given here. The $\boldsymbol{\sigma}$-terms in Eq. (65) imply a coupling of the orbital angular momentum with the spin $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. There is also a term proportional to the hyperbolic unit which has a different parity. The wave function $\psi(x)$ therefore has a similar structure as the Dirac spinor

$$\psi(x) = \varphi_l(x) + j\chi_{l'}(x), \tag{77}$$

with

$$l' = \begin{cases} l-1, & \text{for } l = j+1/2, \\ l+1, & \text{for } l = j-1/2. \end{cases} \tag{78}$$

The eigenvalue of the coupled spin operator should not be confused with the hyperbolic unit in this equation. Since relativistic single particle potentials depend on the relative time, the energy of the single particle states is not a conserved quantity. Therefore, the following eigenvalue problem has to be considered

$$M^2\psi(x) = m^2\psi(x), \tag{79}$$

where the solutions $\psi_i(x)$ are eigenstates of the mass operator with the quantum numbers $i = (njlm_j)$. Given the model potential of Eq. (75) the eigenvalues m_i^2 of this equation should be close to the spectrum of the Dirac equation with a non-relativistic $1/|\boldsymbol{x}|$ central potential. For the ground state $GS = (1\frac{1}{2}0m_j)$ one can therefore expect to find approximately

$$m_{GS}^2 \approx 1 - Z^2\alpha^2. \tag{80}$$

A detailed solution of this problem is beyond the scope of this work.

12. Summary

The representation of relativistic quantum physics in terms of mathematical structures is not unique. The relationships and isomorphisms between different representations can be understood within Clifford algebras as the common underlying mathematical framework. The most popular representation of relativistic physics is based on the Dirac algebra $\boldsymbol{R}_{1,3}$. However, the Majorana algebra $\boldsymbol{R}_{3,1}$, or the $\boldsymbol{R}_{3,0}$ algebra suggested by Baylis, provide frameworks that can be used for relativistic calculations as well.

The three-dimensional universal complex Clifford algebra $\bar{\boldsymbol{C}}_{3,0}$ is proposed for an alternative representation of relativistic quantum physics. The Baylis algebra is isomorphic to the subset of $\bar{\boldsymbol{C}}_{3,0}$ considered in this work. The structural difference appears in the shape of the hyperbolic unit. The full structure of the complex $\bar{\boldsymbol{C}}_{3,0}$ Clifford algebra provides sixteen real dimensions, the same number as the Dirac algebra, which is in contrast to the eight-dimensional Baylis algebra.

Appendix A. Relationship to other representations

The algebra introduced in Eq. (6) seems to be identical to the $\boldsymbol{R}_{3,0}$ paravector algebra of Baylis [34,35]. Especially, if one considers the corresponding anti-involutions in Table 1 for the basis elements e_i. However, Baylis identifies the basis elements e_i directly with the Pauli algebra. In his approach the Pauli algebra therefore has different transformation properties under the anti-involutions than in the approach presented here.

The hyperbolic algebra can be represented also in terms of quaternions as

$$e_\mu = (1, ijq_i), \tag{A.1}$$

where $q_i \in \boldsymbol{Q}$ denote the basis elements of the quaternion algebra. The three-dimensional vector symbol $\boldsymbol{x}$ has been represented as $x^i\sigma_i$, because the Pauli algebra is the one most familiar to physicists. It could be represented also as x^ie_i or as x^iq_i. It is possible to change from one picture to the other if $j\sigma_i$ is replaced by e_i or ijq_i including a redefinition of $\boldsymbol{x}$.

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